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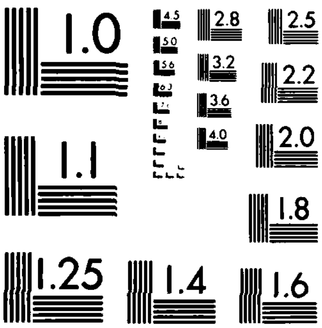

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THE h, p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD IN 1 DIMENSION

PART 3. THE ADAPTIVE h-p VERSION

by

W. Gui and I. Babuska

June 1985

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The paper is the third and final part in the series of three devoted to the detailed analysis of the three basic versions of the finite element method in one dimension. The first part [1] analyzed the p-version, the second part [2 concentrated on the h and h-p version and the present third part addresses the adaptive h-p version.

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Defense Logistics Agency
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Gentlemen:

Enclosed please find fifteen (15) copies each of the following technical reports which were sponsored in part by the Office of Naval Research under Contract N00014-85-K-0169:

Technical Note BN-1036 - The h, p and h-p Versions of the Finite Element Method in 1 Dimension. Part 1. The Error Analysis of the p-Version.

Technical Note BN-1037 - The h, p and h-p Versions of the Finite Element Method in 1 Dimension. Part 2. The Error Analysis of the h and h-p Versions.

Technical Note BN-1038 - The h, p and h-p Versions of the Finite Element Method in 1 Dimension. Part 3. The Adaptive h-p Version.

Sincerely,

Ivo Babuska
Professor

IB:aa

Enclosures (45)

THE h, p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD IN 1 DIMENSION

PART 3. THE ADAPTIVE h-p VERSION

W. Gui* and I. Babuška**

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ABSTRACT

The paper is the third and final part in the series of three devoted to the detailed analysis of the three basic versions of the finite element method in one dimension. The first part [1] analyzed the p-version, the second part [2] concentrated on the h and h-p version, and the present third part addresses the adaptive h-p version.

1. INTRODUCTION

This paper is the third and final part in the series of three which analyzes the h, p and h-p versions of the finite element method in one dimensional setting. It has been shown in Part 1 and 2 that the selection of the mesh and degree of elements is essential for the performance of the method. We have shown that the proper selection of the h-p version leads to the exponential rate of convergence while the h-version with improper mesh, e.g. uniform mesh, gives very low algebraic rate when a singularity is present. The adaptive approaches are essential for solving complex problems, because the structure of the solution is not known a-priori.

In recent years the adaptive methods came to be in the focus of interest. Various papers, see e.g. [3] [4] [5], address the question of adaptive approaches in the Finite Element Method. In two dimensional adaptive research code FEARS (see [6]) and PLTMG (see [7]) are available. Both codes deal with the h-version and linear $(p = 1)$ elements. However, there is no adaptive h-p version code and only little work has been done addressing this question. See [8] [9].

In this paper we analyze a theoretical frame of the adaptive h-p version and based on it we provide concrete algorithm for the one dimensional problem. It is proven that in the case that the solution has x^{α}-type singularity, the adaptive algorithm give an exponential rate of convergence, very close to the optimal one analyzed in the second part of the paper.

We expect that the principles used here in the one dimensional setting could be successfully applied also in the higher dimensional case. Although above we used the notion of an adaptive approach in broad sense, we will later distinguish in a more precise setting between feedback and

adaptive approach (cf. [10] [11] [12] [13]). By feedback approach we understand the approach when previous (computed) information are sequentially used. The adaptive approach is in a feedback which has well defined optimal properties. The distinction between the feedback and adaptive approach is often worthwhile in a more precise analysis.

We will develop in this paper an abstract frame of the adaptive approach and its theory. We are concerned here only with the convergence and its rate in the energy norm. Section 2 focusses on the algorithm, Section 3 deals with its convergence, and Section 4 analyzes its rate of convergence. Section 5 gives some numerical and a short discussion of implementational aspects. Section 6 summarizes the major properties of the three basic versions of the FEM.

2. THE ABSTRACT SETTING OF THE ADAPTIVE h-p VERSON ALGORITHM OF THE FINITE ELEMENT METHOD.

First, we will make some definitions. A <u>mesh</u> Δ is a partition of interval $[0,1]$. For convenience, a mesh Δ may be regarded as a set of nodal points or a set of non-overlapping closed intervals, the union of which is $[0,1]$. The number of intervals contained in a mesh Δ is called the <u>cardinality</u> of a mesh is denoted by $m(\Delta)$. To each mesh interval I_i^Δ we assign a positive integer p_i^Δ, which is called the <u>degree</u> of the mesh interval. These degrees constitute the degree vector $\underline{p}^\Delta$. The superscript Δ indicates its relation to the mesh Δ; if there is no confusion it will be often omitted.

<u>Definition 1</u>. $\mathcal{S}$ is the set of mesh-degree combinations. Its element

$$\Sigma = (\Delta, \underline{p}^\Delta)$$

is called the <u>pair</u>, where Δ is a mesh and $\underline{p}^{\Delta}$ is its associated degree vector.

Σ is also regarded as a set of the pairs (I,p) where $I \in \Delta$ and p is the degree of I.

<u>Definition 2</u>. Let $\Sigma \in \mathcal{S}$, the <u>number of degrees of freedom of</u> Σ <u>is</u>

$$(2.1) \qquad N \equiv \deg(\Sigma) = \sum_{i=1}^{m(\Delta)} p_i^{\Delta}.$$

We can make $\mathcal{S}$ a partially ordered set by defining the following partial ordering on $\mathcal{S}$.

<u>Definition 3</u>. Let $\Sigma_1, \Sigma_2 \in \mathcal{S}$. $\Sigma_1 = (\Delta,\underline{p}^1)$, $\Sigma_2 = (\Delta_2,\underline{p}^2)$, then $\Sigma_1 < \Sigma_2$ if and only if:

1) Δ_2 is a <u>refinement</u> of Δ_1, i.e., as sets of nodal points one has

$$\Delta_1 \subseteq \Delta_2$$

and we will write

$$\Delta_1 < \Delta_2;$$

2) $\underline{p}^2$ is a <u>refinement</u> of $\underline{p}^1$, i.e., $\Delta_1 < \Delta_2$, and if $I_i^{(1)} \in \Delta_1$, $I_j^{(2)} \in \Delta_2$, $I_i^{(2)} \subseteq I_j^{(1)}$, then $p_i^{(1)} < p_j^{(2)}$. In this case we write

$$\underline{p}^{(1)} < \underline{p}^{(2)}.$$

($p_i^{(1)}$ and $p_j^{(2)}$ are the degrees associated to the intervals $I_i^{(1)}$, $I_j^{(2)}$ resp.)

We will call Σ_2 a <u>refinement</u> of Σ_1.

<u>Definition 4</u>. The <u>local error function</u>

$$E([a,b],p)$$

is a non-negative real valued function defined on the set

$$\{0 \leq a \leq b \leq 1, \quad p \geq 1, \quad p \text{ is integer}\},$$

which satisfies the following hypotheses:

(E1) $E([a,b],p)$ is continuous in a, b. It is non-increasing in a and p, and non-decreasing in b.

(E2) $E([a,a],p) = 0$ for $0 \leq a \leq 1$, $p \geq 1$.

(E3) (p-approximability). For any fixed $[a,b] \subseteq [0,1]$

$$\lim_{p\to\infty} E([a,b],p) = 0.$$

For some given λ, $1 \leq \lambda < \infty$:

(E4) (reverse sub-additivity). If $c \in [a,b]$, $p \geq 1$, then

$$\{E([a,c],p)^\lambda + E([c,b],p)^\lambda\}^{1/\lambda} \leq E([a,b],p).$$

(E5) (h-approximability). Let $\{\Delta_n\}_{n=1}^\infty$ be a sequence of meshes for which

$$\lim_{n\to\infty} \max_{I\in\Delta_n} \{|I|\} = 0$$

then

$$\lim_{n\to\infty} \{\sum_{I\in\Delta_n} E(I,1)^\lambda\}^{1/\lambda} = 0.$$

(if $\lambda = \infty$, $a_i \geq 0$, we define $\{\sum_{i=1}^{n} a_i^{\lambda}\}^{1/\lambda} = \max_{1 \leq i \leq m} a_i$).

The number λ is called the <u>index</u> of the local error function.

<u>Remark 1</u>. These hypotheses are very natural if we consider the error function to be the local error of the best approximation in a certain function space. For example, let $u \in L_2(0,1)$ and

$$E([a,b],p) = \inf_{v \in \mathbf{P}_{p-1}} \|u-v\|_{L_2(a,b)},$$

then it is easy to check that all hypotheses are satisfied for $\lambda = 2$.

We have the following simple corollaries which follow immediately from the definition:

<u>Corollary 1</u>. If $I_1 \subseteq I_2$, then

$$E(I_1,p) \leq E(I_2,p) \qquad (\forall p \geq 1). \tag{2.2}$$

<u>Corollary 2</u>. If $a = x_0 < x_1 < \cdots < x_k = b$, $p \geq 1$, then

$$\{\sum_{i=1}^{k} E([x_{i-1},x_i],p)^{\lambda}\}^{1/\lambda} \leq E([a,b],p). \tag{2.3}$$

<u>Proof</u>. (2.2) follows from (E1). (2.3) follows from (E4) by induction. ∎

<u>Definition 5</u>. A <u>local error indicator</u> $e([a,b],p)$ is a non-negative real valued function defined on the same set as $E([a,b],p)$. And there is a constant $0 < C \leq 1$, independent of a, b and p, that

$$Ce([a,b],p) \leq E([a,b],p) \leq C^{-1}e([a,b],p). \tag{2.4}$$

It is obvious that for the local error indicate we have:

Corollary 3. The following properties hold:

(E2)′ $e([a,a],p) = 0$ for all $0 \leq a \leq 1$, $p \geq 1$.

(E3)′ $\lim_{p\to\infty} e([a,b],p) = 0$ for any $[a,b] \subseteq [0,1]$.

(E5)′ Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of meshes, $1 \leq \lambda \leq \infty$, and

$$\lim_{n\to\infty} \max_{I\in\Delta_n} \{|I|\} = 0$$

then

$$\lim_{n\to\infty} \{\sum_{I\in\Delta_n} e(I,1)^{\lambda}\}^{1/\lambda} = 0.$$

(λ is the index of local error function).

Definition 6. The global error based on the pair $\Sigma = (\Delta,\underline{p})$ is given by

$$E_{\lambda}(\Sigma) = \{\sum_{I_i\in\Delta} E(I_i,p_i)^{\lambda}\}^{1/\lambda} \tag{2.5}$$

where $1 \leq \lambda \leq \infty$, p_i is the degree of I_i.

Similarly, the global error estimator based on the pair Σ is given by

$$e_{\lambda}(\Sigma) = \{\sum_{I_i\in\Delta} e(I_i,p_i)^{\lambda}\}^{1/\lambda}. \tag{2.6}$$

Clearly we have

Corollary 4. Let C be given (2.4), $1 \leq \lambda \leq \infty$, Σ be any pair, then

$$Ce_{\lambda}(\Sigma) \leq E_{\lambda}(\Sigma) \leq C^{-1}e_{\lambda}(\Sigma). \tag{2.7}$$

We now define the feedback h-p version algorithm (we will call it

<u>the algorithm</u> below).

First, let the local error indicator $e([a,b],p)$ be given. We will divide the intervals of any mesh into two categories, called the <u>h-intervals</u> and the <u>p-intervals</u>. We will also say that an interval is of <u>h-type</u> or <u>p-type</u>. The type of an interval is defined by

<u>Definition 7</u>. Let $0 \le \gamma < \infty$ be given. Let $\Sigma = (\Delta, \underline{p})$ be a given pair, $I \in \Delta$ and p the degree of I, let

$$R = \frac{e(I,p+1)}{e(I,p)}, \qquad (R = 0 \text{ if } e(I,p) = 0).$$

Then if $R \ge \gamma$, I is said to be an h-interval; if $R < \gamma$, I is said to be a p-interval (with respect to γ and Σ).

The number γ is called the <u>type-parameter</u>. Usually, we are interested in the case $0 < \gamma < 1$.

The feedback algorithm is now defined in a recurrent way:

Let $0 < \theta < 1$ be a given number, called the <u>refinement-parameter</u> $\Sigma_n = (\Delta_n, \underline{p}^n)$, and

$$e_{max}^{(n)} = \max_{1 \le i \le m(\Delta_n)} e(I_i^{(n)}, p_i^{(n)})$$

with $(I_i^{(n)}, p_i^{(n)}) \in \Sigma_n$. (The interval $I_j^{(n)} \in \Delta_n$ on which $e_{max}^{(n)}$ is realized will be called the <u>critical interval</u>.) Denoting

$$e_i^{(n)} = e(I_i^{(n)}, p_i^{(n)})$$

then all intervals $I_i^{(n)} \in \Delta_n$ with $e_i^{(n)} < \theta \cdot e_{max}^{(n)}$ will stay the same in the mesh Δ_{n+1} of the new pair Σ_{n+1} and the degrees $p_i^{(n)}$ will be also kept. If for some interval $I_i^{(n)} \in \Delta_n$ on which $e_i^{(n)} \ge \theta \cdot e_{max}^{(n)}$,

then there are two cases:

1) if $I_i^{(n)}$ is an h-interval, then it will be bisected, and the degree $p_i^{(n)}$ will be inherited by both the bisected intervals;

2) If $I_i^{(n)}$ is a p-interval, then $I_i^{(n)}$ remains an interval of Δ_{n+1} but its degree is assigned to be $p_i^{(n)} + 1$.

We write

$$\Sigma_{n+1} = T(\Sigma_n) \equiv T(\Sigma_n, e, \gamma, \theta)$$

where e stands for the local error indicator, γ the type-parameter and θ the refinement-parameter.

Definition 8. The above rule T of constructing the pair Σ_{n+1} from an existing pair Σ_n is called the transition operator of the algorithm. The subset $\{\Sigma_n\}_{n=0}^{\infty}$ of S, where $\Sigma_n = T(\Sigma_{n-1})$ for $n = 1,2,\ldots,$ is called a trajectory of the transition operator.

It is obvious that we have

Corollary 5. A trajectory $\{\Sigma_n\}$ is a monotone increasing sequence in S, namely, $\Sigma_n < \Sigma_{n+1}$ for all $n = 0,1,2,\ldots$.

Remark 2. There are two degenerated cases for the algorithm: if $\gamma = 0$, then all intervals are of h-type and this algorithm gives a feedback h-version. If $\gamma = \infty$, then all intervals are of p-types and will never be bisected; in this case we will obtain a feedback p-version.

Remark 3. We are speaking about feedback algorithm because the current information steers the flow of algorithm. Often such algorithm is called also adaptive (see e.g. [10], pp. 49-50, [11]). We shall distinguish

between a feedback algorithm and an adaptive algorithm in the sense that the adaptive algorithm is a feedback having well defined optimality properties (see [12], [13], [4]). In the next section we will prove that the algorithm is convergent and hence it is adaptive with respect to the convergence measure.

3. THE CONVERGENCE OF THE ALGORITHM

Definition 9. If for any trajectory $\{\Sigma_n\}_{n=0}^{\infty}$ of the transition operator

$$\lim_{n\to\infty} E_\lambda(\Sigma_n\} = 0,$$

then the algorithm is said to be convergent.

We will prove the feedback algorithm defined above is convergent. First we observe that the (E5) implies:

Lemma 1. Let $\{\pi_n\}_{n=0}^{\infty}$ be a sequence of sets of non-overlapping closed intervals (not necessarily covering the entire interval $[0,1]$). If

$$\lim_{n\to\infty} \max_{I\in\pi_n} \{|I|\} = 0,$$

then

$$\lim_{n\to\infty} \{ \sum_{I\in\pi_n} E(I,1)^\lambda\}^{1/\lambda} = 0.$$

Proof. There is Δ_n such that

1) as sets on non-overlapping closed intervals,

$$\pi_n \subseteq \Delta_n.$$

2) $$\max_{I\in\Delta_n} \{|I|\} = \max_{I\in\pi_n} \{|I|\}.$$

Applying (E5) to the mesh sequen~e $\{\Delta_n\}_{n=0}^{\infty}$, we obtain using (E5)

$$\{\sum_{I\in\pi_n} E(I,1)^{\lambda}\}^{1/\lambda} \leq \{\sum_{I\in\Delta_n} E(I,1)^{\lambda}\}^{1/\lambda} \to 0$$

as $n \to \infty$.

As consequences, it is easy to obtain:

Corollary 6. If $\{I^{(n)}\}_{n=0}^{\infty}$ is a sequence of closed intervals, $p^{(n)} \geq 1$ and $\lim_{n\to\infty} |I^{(n)}| = 0$, then

$$\lim_{n\to\infty} e(I^{(n)},p^{(n)}) = 0.$$

Corollary 7.

Let $\{\pi_n\}_{n=0}^{\infty}$ be given as in Lemma 1. Assign $I_i^{(n)} \in \pi_n$ an integer $p_i^{(n)} \geq 1$ and assume that

$$\lim_{n\to\infty} \max_{I_i^{(n)}\in\pi_n} \{|I_i^{(n)}|\} = 0$$

then

$$\lim_{n\to\infty} \{\sum_{I_i^{(n)}\in\pi_n} e(I_i^{(n)},p_i^{(n)})^{\lambda}\}^{1/\lambda} = 0.$$

We now prove

Theorem 1. The feedback algorithm is convergent.

Proof. Let $\{\Sigma_n\}_{n=0}^{\infty}$ be a trajectory of the transition operator $\mathcal{T}$. By Corollary 5

(3.1) $$\Sigma_0 < \Sigma_1 < \Sigma_2 \cdots .$$

First we show that

(3.2) $$\lim_{n\to\infty} \max_{I^{(n)}\in\Delta_n} \{E(I^{(n)},p^{(n)})\} = 0$$

where Δ_n is the mesh of Σ_n, and $p^{(n)}$ is the degree of $I^{(n)} \in \Delta_n$. By (2.4) it is enough to show that

(3.3) $$\lim_{n\to\infty} \max_{I^{(n)}\in\Delta_n} \{e(I^{(n)},p^{(n)})\} = 0.$$

Suppose this is not true, then there is a subsequence $\{n_k\}$ and a number $\varepsilon > 0$ such that

(3.4) $$e_{max}^{(n_k)} = \max_{I^{(n_k)}\in\Delta_{n_k}} \{e(I^{(n_k)},p^{(n_k)})\} > \varepsilon.$$

Furthermore, let $I_c^{(n_k)} \in \Delta_{n_k}$ be the critical intervals, i.e. $e_{max}^{(n_k)}$ is realized on $I_c^{(n_k)}$. Corollary 6 implies that there is another subsequence of $\{n_k\}$, we still denote it by $\{n_k\}$, such that

$$|I_c^{(n_k)}| > h$$

for some $h > 0$, for otherwise $e_{max}^{(n_k)} \to 0$. Since $\Sigma_{n_k} < \Sigma_{n_{k+1}}$, two intervals $I_c^{(n_k)}$ and $I_c^{(n_{k+1})}$ are either non-overlapping or $I_c^{(n_k)} \supseteq I_c^{(n_{k+1})}$. Because $I_c^{(n_k)} \subseteq [0,1]$, there can only be a finite number of non-overlapping intervals with length $> h > 0$. Thus we conclude there is again a subsequence of $\{n_k\}$, we again denote it by

$\{n_k\}$, such that for some $k > 0$

$$I^{(n_k)} \supseteq I_c^{(n_{k+1})} \supseteq I_c^{(n_{k+2})} \supseteq \cdots .$$

Each $I_c^{(n_{k+1})}$ is either a result of several bisections of $I_c^{(n_k)}$, or $I_c^{(n_{k+1})} = I_c^{(n_k)}$. Because by our assumption $|I_c^{(n_k)}| \geq h$, there is k' such that $I_c^{(n_k)}$ is never bisected for $k > k'$ and hence $I_c^{(k')}$ is the p-interval. Because $I_c^{(n_k)}$ are critical intervals, we conclude that

$$\lim_{k\to\infty} p_c^{(n_k)} = +\infty$$

where $p_c^{(n_k)}$ is the degree of $I_c^{(n_k)}$. By (E3)′, $\lim_{k\to\infty} e_{max}^{(n_k)} = \lim_{k\to\infty} e(I_c^{(n_k)}, p_c^{(n_k)}) = 0$. This contradicts (3.4).

For $\lambda = \infty$ theorem follows immediately.

Assume now that $1 \leq \lambda < \infty$. Since (3.1) implies

$$E_\lambda(\Sigma_0) \geq E_\lambda(\Sigma_1) \geq E_\lambda(\Sigma_2) \geq \cdots$$

(E1) and (E4), it suffices to show that for each $\varepsilon > 0$, there is $N(\varepsilon) \geq 0$ such that

$$E_\lambda(\Sigma_{N(\varepsilon)}) < \varepsilon.$$

For each $h > 0$ we define

$$\Sigma^h = \{I \in \bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} \Delta_n;\ |I| \geq h\}$$

where Δ_n are regarded as sets of intervals. Then for each h there is $N_0(h) > 0$ such that if $n > N_0(h)$, then

$$\pi_n = \Delta_n \setminus \Sigma^h$$

contains no interval I with $|I| > h$. Let $h_j \downarrow 0$ and choose corresponding $n_j = N_0(h_j)$ to be such that $n_j \uparrow \infty$. Then the sequence $\{\pi_{n_j}\}_{j=0}^{\infty}$ satisfies the condition of Corollary 7 and thus

$$\lim_{j\to\infty} \left\{ \sum_{I_i^{(n_j)} \in \pi_{n_j}} e(I_i^{(n_j)}, p_i^{(n_j)})^{\lambda} \right\}^{1/\lambda} = 0.$$

Therefore there is $j_0 = j(\varepsilon)$ such that

$$(3.5) \qquad \left\{ \sum_{I_i^{(n_{j_0})} \in \pi_{n_{j_0}}} E(I_i^{(n_{j_0})}, p_i^{(n_{j_0})})^{\lambda} \right\}^{1/\lambda} < \varepsilon/2^{1/\lambda}.$$

If $\Sigma^{h_{j_0}} = \emptyset$, the proof is finished. Suppose $\Sigma^{h_{j_0}} \neq \emptyset$. Let M be the number of intervals contained in $\Sigma^{h_{j_0}}$. By the first part of the proof, there is $j_1 > j_0$ such that

$$E(I_i^{(n_{j_1})}, p_i^{(n_{j_1})}) < \varepsilon/(2M)^{1/\lambda}$$

for all $I_i^{(n_{j_1})} \in \Sigma^{h_{j_0}}$ (recall that $\Sigma^{h_{j_0}}$ contains the intervals which are never bisected againt). Therefore we have

$$(3.6) \qquad \left\{ \sum_{I_i^{(n_{j_1})} \in \Sigma^{h_{j_0}}} E(I_i^{(n_{j_1})}, p_i^{(n_{j_1})})^{\lambda} \right\}^{1/\lambda} < \varepsilon/2^{1/\lambda}.$$

Furthermore, the intervals contained in $\Delta_{n_{j_1}} \setminus \Sigma^{h_{j_0}}$ are either inherited from $\pi_{n_{j_0}}$ or the results of several times bisections of the intervals of $\pi_{n_{j_0}}$, (3.5) and (E1), (E4) imply that

$$(3.7) \qquad \left\{ \sum_{I_i^{(n_{j_1})} \in \Delta_{n_{j_1}} \setminus \Sigma_{h_{j_1}}} E(I_i^{(n_{j_1})}, p_i^{(n_{j_1})})^{\lambda} \right\}^{1/\lambda} < \varepsilon/2^{1/\lambda}.$$

and (3.6), (3.7) give

$$E_\lambda(\Sigma_{n_{j_1}}) < \varepsilon.$$

Thus the proof is complete. ∎

<u>Remark 4</u>. Since in the proof we did not use any information of the parameter γ, we have shown a convergence of all feedback h, p and h-p version algorithms.

<u>Remark 5</u>. We introduced a family of feedback algorithms which create trajectories $\{\Sigma_n\}$. We define the performance measure μ_0 of the algorithm so that if $E_\lambda(\Sigma_n) \to 0$ for any trajectory $\{\Sigma_n\}$ of the algorithm, then $\mu_0 = 1$, otherwise $\mu_0 = 0$; and we define the optimal performance measure to be with maximal value (1 in this case). Thus we can say that our feedback algorithm is adaptive with respect to the performance measure μ_0 (called convergence measure, see [12], pp. 7-8).

4. THE RATE OF CONVERGENCE OF THE ALGORITHM

In order to study the rate of convergence of the algorithm it is necessary to have more knowledge on the local error function. Motivated by the results of Part 1 [1], we will study the algorithm on the class of local error functions which satisfy the following hypotheses:

(A1) There is a point $\xi \in [0,1]$, called the <u>singular point</u>. If $\xi \in I \subseteq [0,1]$, then

$$E(I,1) \leq C_0 |I|^{\sigma},$$

where $\sigma > 0$, $C_0 > 0$ are constants independent of I.

(A2) There is a non-increasing continuous function $\phi : (0,\infty) \to (0,1)$ with

$$\lim_{t \to 0} \phi(t) = 1$$

$$\lim_{t \to \infty} \phi(t) = 0$$

such that for any $\varepsilon > 0$, $p \geq 1$, $I \subseteq [0,1]$, $\xi \notin I$, and $t = \frac{\text{dist}(\xi,I)}{|I|} \geq \varepsilon$, there is $C(\varepsilon) > 0$ such that

$$E(I,p) \leq C(\varepsilon)[\phi(t)]^{p}.$$

(A3) If $\xi \notin I$, $I \subseteq [0,1]$, then there is $r = r(I) > 0$, such that $E(I,p) > 0$, $p \geq 1$

$$\kappa r \leq \frac{E(I,p+1)}{E(I,p)} \leq r$$

the constant $0 < \kappa \leq 1$ is independent of I, and p.

Observe that the hypotheses (A1), (A2) are the characterisics of the best L_2-approximation error of the analytic function with an x^{α}-type

singularity at ξ. In Part 1 [1] we have shown that if the local error function $E(I,p)$ represents the L_2-error, then $\sigma = \alpha + \frac{1}{2}$, and $\phi(t) = 1/(1+2t+2\sqrt{t(1+t)}\,)$. The results of Part 1 also show that for the function $(x-\xi)_+^\alpha$ the hypothesis (A3) is satisfied.

Lemma 2. Let the local error function $E(I,p)$ satisfy (A1) ~ (A3), $e(I,p)$ be the local error indicator of $E(I,p)$, then $e(I,p)$ also satisfies (A1) ~ (A3) with different constants. More precisely, we have:

(A1)′ If $\xi \in I \subseteq [0,1]$, then

$$e(I,1) \leq C_0'|I|^\sigma,$$

where σ is the same as in (A1), $C_0' > 0$ independent of I.

(A2)′ If $\xi \notin I$, $I \subseteq [0,1]$, $t = \frac{\mathrm{dist}(\xi,I)}{|I|} \geq \varepsilon > 0$, then there is $C'(\varepsilon) > 0$ for which

$$e(I,p) \leq C'(\varepsilon)[\phi(t)]^p$$

where $p \geq 1$, ϕ is the same as in (A2).

(A3)′ If $\xi \notin I$, $I \subset [0,1]$, then there is $r' = r'(I) > 0$, such that

$$\kappa' r' \leq \frac{e(I,p+1)}{e(I,p)} \leq r'$$

with $p \geq 1$, $0 < \kappa' \leq 1$, κ' is independent of I and p.

Proof. (A1)′, (A2)′ are obvious. Suppose (A3) holds, then by (2.4) we have

$$\frac{e(I,p+1)}{e(I,p)} \leq \frac{C^{-1}E(I,p+1)}{CE(I,p)} \leq C^{-2}r$$

$$\frac{e(I,p+1)}{e(I,p)} \geq \frac{CE(I,p+1)}{C^{-1}E(I,p)} \geq C^2\kappa r$$

where $r = r(I) > 0$, $0 < C \leq 1$, $0 < \kappa \leq 1$. Let

$$r' = r'(I) = C^{-2} r(I),$$

$$\kappa' = C^4 \kappa,$$

and (A3)′ is satisfied. ∎

By the hypothesis (E1) it is easy to see that (A1) thus (A1)′ may be extended to $p \geq 1$:

Corollary 8. If $\xi \in I \subseteq [0,1]$, $p \geq 1$, then

$$E(I,p) \leq C_0 |I|^\sigma$$

$$e(I,p) \leq C_0' |I|^\sigma$$

hold uniformly with respect to I and p. ∎

Lemma 3. Let $E(I,p)$ satisfy (A2) and (A3). Then we have

$$r \leq \kappa^{-1} q \tag{4.1}$$

where r, κ are defined in (A3), $q = \phi(t)$ is given in (A2) and $t = \frac{\operatorname{dist}(\xi,I)}{|I|}$.

Proof. By (A3) there is r such that

$$\kappa r \leq \frac{E(I,p+1)}{E(I,p)} \leq r.$$

This implies

$$E(I,p) \geq C(\kappa r)^p$$

with $p \geq 1$, C being a constant independent of p. Comparing with (A2) we then obtain for any $p \geq 1$

$$\kappa r \leq q = \phi(t)$$

with $t = \frac{\mathrm{dist}(\xi,I)}{|I|}$. Thus

$$r \leq \kappa^{-1}q. \qquad \blacksquare$$

<u>Corollary 9</u>. If $E(I,p)$ satisfies (A2), (A3)$'$, then

(4.2) $$r' \leq k'^{-1}q$$

with r', κ' given in Lemma 2, $q = \phi(t)$ as before.

<u>Proof</u>. Obvious. $\blacksquare$

<u>Lemma 4</u>. Let $E(I,p)$ satisfy (A2), (A3)$'$, $0 < \gamma < 1$ is the type-parameter (as defined in Section 2), and $\delta > 0$ be determined such that

$$\phi(\delta) = \kappa'\gamma_1$$

where $0 < \gamma_1 < \gamma < 1$ and κ' is the constant in (A3)$'$. If $x \in I \subseteq [0,1]$,

$x \neq \xi$, and

(4.3) $$|I| \leq \frac{|x-\xi|}{1+\delta},$$

then

$$R = \frac{e(I,p+1)}{e(I,p)} < \gamma$$

where $p \geq 1$. Therefore, I is a p-interval.

<u>Proof</u>. Recall $0 < \kappa' \leq 1$ (see Lemma 2). Since $0 < \gamma_1 < \gamma < 1$, there exists $\delta > 0$ satisfying $\phi(\delta) = \kappa'\gamma_1$. (4.3) implies

$$t = \frac{\mathrm{dist}(\xi,I)}{|I|} \geq \frac{|x-\xi|-|I|}{|I|} \geq \delta,$$

therefore $\phi(t) \leq \phi(\delta) = \kappa'\gamma_1$, and by (A3)′ and (4.2)

$$R = \frac{e(I,p+1)}{e(I,p)} \leq (\kappa')^{-1}(\kappa'\gamma_1) = \gamma_1 < \gamma,$$

thus I is a p-interval. ∎

We now study how the algorithm does if the hypotheses (A1) - (A3) are satisfied. For simplicity we assume the trajectory $\{\Sigma_\nu\}_{\nu=0}^{\infty}$ starts with the mesh $\Delta_0 = \{[0,1]\}$, and $\underline{p}^0 = (p_0)$, $p_0 \geq 1$. In this case, any meshes of the trajectory can only contain the interval of the form

$$\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right], \qquad 1 \leq k \leq 2^n, \quad n = 0,1,2,\ldots \ . \tag{4.4}$$

<u>Definition 10</u>. An interval of the form (4.4) is called a <u>binary interval of level</u> n.

<u>Lemma 5</u>. Let the local error function satisfy the hypotheses (A2) - (A3). Let γ be the type-parameter $0 < \gamma < 1$. If $x \in [0,1]$, $x \neq \xi$, then the algorithm will generate a binary interval I such that $x \in I$, I is a p-interval and it remains to be a p-interval in the further process.

<u>Proof</u>. By the proof of Theorem 1 we have shown (without (A2) - (A3))

$$\lim_{\nu\to\infty} \max_{I_j^{(\nu)}\in\Delta_\nu} \{e(I_j^{(\nu)},p_j^{(\nu)})\} = 0.$$

Therefore, if $x \in I_j^{(\nu)} \in \Delta_\nu$, $x \neq \xi$ and $e(I_j^{(\nu)},p_j^{(\nu)}) > 0$, in the further process either $I_j^{(\nu)}$ will be bisected or $p_j^{(\nu)}$ will go up. By

Lemma 4 the bisection must stop if the size of the interval is small enough, in other words, there is ν_0 such that if $x \in I_j^{(\nu_0)}$, then $I_j^{(\nu_0)}$ is a p-interval and lies in all Δ_ν, $\nu > \nu_0$. ∎

Lemma 6. Let the local error function satisfy (A2) - (A3), and $\delta > 0$ be defined in Lemma 4. Suppose that the pair Σ is generated by the algorithm and it has its smallest interval which is of level n, then the total number of intervals in the mesh Δ of Σ is bounded by

$$M = 2(n+1)(L+1) + 1 \tag{4.5}$$

where $L = [1+\delta]$.

Proof. By Lemma 4, if $x \in I$, $x \neq \xi$ and

$$|I| \leq \frac{1}{1+\delta} |x-\xi|,$$

then I is permanently a p-interval (i.e., it remains as a p-interval in further process). This is true if

$$|I| \leq \frac{1}{1+L} |x-\xi| \tag{4.6}$$

where $L = [1+\delta]$.

If this interval is of level k, then $|I| = \frac{1}{2^k}$ and

$$|x-\xi| \geq \frac{1+L}{2^k}. \tag{4.7}$$

Without loss of generality we can consider the interval which is right to ξ. Denoting $d = \mathrm{dist}(\xi, I)$, (4.7) implies that for each k there are no h-intervals of level k with

$$d \geq \frac{1+L}{2^k}. \tag{4.8}$$

Because each p-interval, except $[0,1]$, is obtained by bisecting an h-interval, the above implies that there are no p-intervals of level k with

$$d \geq \frac{1+L}{2^{k-1}} + \frac{1}{2^k} = \frac{2L+3}{2^k}. \tag{4.9}$$

Now within a distance from ξ between $\frac{1+L}{2^{k+1}}$ and $\frac{1+L}{2^k}$, one can at most put in $[\frac{1+L}{2}]$ h-intervals (of level k) or $(1+L)$ p-intervals. Either way, the number of intervals within above range is bounded by $(1+L)$. This is true for all $0 \leq k \leq n$. Within a distance ranging from 0 to $\frac{1+L}{2^n}$, there can be at most $(1+L)$ intervals since the smallest interval is of level n. Hence the total number of intervals to one side of ξ will not exceed $(n+1)(1+L)$. Including the interval containing ξ, total number of intervals in the mesh then will not exceed

$$M = 2(1+L)(n+1) + 1. \qquad \blacksquare$$

<u>Lemma 7</u>. Let the local error function satisfy (A1) - (A3), $0 < \gamma < 1$, $0 < \theta < 1$, and let $\{\Sigma_\nu\}_{\nu=0}^{\infty}$ be a trajectory starting with $\Sigma_0 = ([0,1],p_0)$, $p_0 \geq 1$. In addition, let the local error indicator $e(I,p)$ satisfy the hypothesis:

(E1)′ if $I, I_1, I_2 \subseteq [0,1]$, $I_1 \subseteq I_2$, $p \geq 1$, then

$$e(I_1,p) \leq e(I_2,p)$$

$$e(I,p+1) \leq e(I,p).$$

Then we have

$$(4.10) \qquad E_\lambda(\Sigma_\nu) \leq C_1[m(\Delta_\nu)]^{1/\lambda} e^{-C_2\left[\frac{\deg(\Sigma_\nu)}{m(\Delta_\nu)}\right]}$$

where Δ_ν is the mesh of Σ_ν, $m(\Delta_\nu)$ is the number of intervals in Δ_ν, $\deg(\Sigma_\nu)$ is given by (2.1), and C_1, C_2 are positive constants independent of ν.

Proof. First we claim that there is a constant C independent of ν such that if $(I,p) \in \Sigma_\nu$, then

$$(4.11) \qquad e(I,p) \leq C\gamma^p.$$

We prove this by induction on $p \geq p_0$. Note that the trajectory is an increasing sequence of pairs

$$(4.12) \qquad \Sigma_0 \leq \Sigma_1 \leq \Sigma_2 \leq \cdots .$$

Let $I_{[p]}$ denote an interval which has degree p. Suppose for some ν, $I_{[p]} \in \Delta_\nu$, and $p \geq p_0$. Let

$$I_{[p_0]} \supseteq I_{[p_0+1]} \supseteq \cdots \supseteq I_{[p-1]} \supseteq I_{[p]}$$

be the sequence of the successsive ancestor intervals of $I_{[p]}$, each of which has corresponding degree as indicated by the subscript. By the hypothesis (E1)′ we have

$$(4.13) \qquad e(I_{[p_0]},p_0) \leq e([0,1],p_0) \equiv C\gamma^{p_0}.$$

We not let $p > p_0$. Suppose

$$e(I_{[p-1]},p-1) \leq C\gamma^{p-1}.$$

If $I_{[p]} = I_{[p-1]}$, then $I_{[p-1]}$ is a p-interval. Thus

$$e(I_{[p]},p) < \gamma e(I_{[p]},p-1)$$

$$= \gamma e(I_{[p-1]},p-1) \leq C\gamma^p.$$

If $I_{[p]} \neq I_{[p-1]}$, then there is $I^*_{[p]} = I^*_{[p-1]}$ such that $I_{[p]} \subseteq I^*_{[p]} \subseteq I_{[p-1]}$ thus by (E1)′

$$e(I_{[p]},p) \leq e(I^*_{[p]},p)$$

$$\leq \gamma e(I^*_{[p-1]},p-1) \leq C\gamma^p.$$

This proves (4.11).

Let $e_{max}^{(\nu)}$ be the maximal local error indicator for Σ_ν. For $0 < \theta < 1$ we define $\hat{p}^{(\nu)}$ by

(4.14) $$C\gamma^{\hat{p}^{(\nu)}-1} = \theta e_{max}^{(\nu-1)} \qquad \nu = 1,2,3,\ldots$$

where C is given in (4.11). By the hypothesis (E1)′ and (4.12) it is easy to see that

$$e_{max}^{(\nu_1)} \geq e_{max}^{(\nu_2)}, \qquad \text{if} \quad \nu_1 \leq \nu_2,$$

and it follows immediately that

(4.15) $$\hat{p}^{(\nu_1)} \leq \hat{p}^{(\nu_2)}, \qquad \text{if} \quad \nu_1 \leq \nu_2.$$

We now claim that

$$p_{max}^{(\nu)} \leq \hat{p}^{(\nu)}, \qquad \text{for all} \quad \nu \geq 1 \tag{4.16}$$

where $p_{max}^{(\nu)}$ is the maximal degree of the intervals of Σ_ν. We will prove this by induction. Observe that (4.11) and (4.14) imply that

$$C\gamma^{p_0} \geq e([0,1],p_0) > \theta e_{max}^{(0)}$$

$$= C\gamma^{\hat{p}^{(1)}-1},$$

thus

$$p_0 \leq \hat{p}^{(1)} - 1.$$

Since $p_{max}^{(1)}$ equals either p_0 or $p_0 + 1$ we obtain

$$p_{max}^{(1)} \leq \hat{p}^{(1)}.$$

Suppose we have

$$p_{max}^{(\nu-1)} \leq \hat{p}^{(\nu-1)}.$$

Now for $p_{max}^{(\nu)}$, either $p_{max}^{(\nu)} = p_{max}^{(\nu-1)}$ or $p_{max}^{(\nu)} = p_{max}^{(\nu-1)} + 1$. In the first case

$$p_{max}^{(\nu)} \leq \hat{p}^{(\nu-1)} \leq \hat{p}^{(\nu)};$$

and in the second case the interval I having degree $p_{max}^{(\nu)}$ was a p-interval of $\Sigma_{\nu-1}$. Hence according to the algorithm we have

$$e(I,p_{max}^{(\nu-1)}) \geq \theta e_{max}^{(\nu-1)} = C\gamma^{\hat{p}^{(\nu)}-1}.$$

By (4.11) this gives

$$p_{max}^{(\nu-1)} \leq \hat{p}^{(\nu)} - 1,$$

thus

$$p_{max}^{(\nu)} \leq \hat{p}^{(\nu)}.$$

Therefore (4.16) is true for all $\nu \geq 1$. From (4.14) and (4.16) we obtain

$$e_{max}^{(\nu)} \leq e_{max}^{(\nu-1)} \leq \frac{C}{\theta\gamma} \cdot \gamma^{p_{max}^{(\nu)}}$$

$$\leq \frac{C}{\theta\gamma} \cdot \gamma^{\frac{\deg \Sigma_N}{m(\Delta)}}.$$

Then

$$e_\lambda(\Sigma_\nu) = \left\{ \sum_{i=1}^{m(\Delta_\nu)} e(I_i^{\Delta_\nu}, p_i^{\Delta_n})^\lambda \right\}^{1/\lambda}$$

$$\leq \frac{C}{\theta\gamma} (m(\Delta_\nu))^{1/\lambda} \gamma^{\frac{\deg \Sigma_\nu}{m(\Delta_\nu)}}$$

and (4.10) follows. In particular $C_2 = \ln \frac{1}{\gamma}$. ■

We now prove the main theorem.

Theorem 2. Let hypotheses (A1) - (A3) and (E1)′ hold, $0 < \gamma < 1$, $0 < \theta < 1$. Let $\{\Sigma_\nu\}_{\nu=0}^{\infty}$ be a trajectory starting with $\Sigma_0 = ([0,1], p_0)$, $p_0 \geq 1$. Then there exist positive constants C_1 and C_2 independent of ν such that

$$E_\lambda(\Sigma_\nu) \leq C_1 e^{-C_2(\sigma \deg \Sigma_\nu)^{1/2}} \tag{4.17}$$

(where $1 \leq \lambda \leq \infty$ is the index of error function (see Definition 4) and

σ is the exponent in hypothesis (A1)).

Proof. First if $m(\Delta_\nu)$ is bounded by a finite number, then by Lemma 7 we will have an estimate $C_1 e^{-C_2 \deg \Sigma_\nu}$ which is better than (4.17). Therefore we can assume $m(\Delta_\nu) \to \infty$ for $\nu \to \infty$.

Suppose that the smallest interval I of Σ_ν has a level n, thus $|I| = 2^{-n}$. Let $e^{(\nu)}_{max}$ be the maximal local error indicator. Furthermore, let J be the parent interval from which I was obtained by bisecting J in Σ_{ν_1}, $\nu_1 \leq \nu - 1$. Therefore J is an h-interval of level $n - 1$, and by (4.8) we must have

$$\text{(4.18)} \qquad \operatorname{dist}(\xi, J) < \frac{L+1}{2^{n-1}}$$

where L is given in Lemma 6.

Let $\hat{I}$ be such an interval that $\xi \in \hat{I}$ and $J \subseteq \hat{I}$, it is easy to obtain $\hat{I}$ with

$$|\hat{I}| \leq \frac{L+1}{2^{n-1}} + \frac{1}{2^{n-1}} = \frac{L+2}{2^{n-1}}.$$

By (A1)´, we obtain

$$I(\hat{I},p) \leq C_0 |\hat{I}|^\sigma \leq C_0 \left(\frac{L+2}{2^{n-1}}\right)^\sigma$$

where p is the degree of J in Σ_{ν_1}, and by (E1)´ we get

$$e(J,p) \leq e(\hat{I},p) \leq C_0 \left(\frac{L+2}{2^{n-1}}\right)^\sigma.$$

Because J was bisected, we must have

$$e(J,p) \geq \theta\, e_{max}^{(\nu_1)}.$$

Using (E1)′ we derive

$$(4.19) \qquad e_{max}^{(\nu)} \leq e_{max}^{(\nu_1)} \leq \theta^{-1} e(J,p) \leq C\left(\frac{1}{2^{\sigma}}\right)^{n}$$

where $C = \theta^{-1} C_0 (2L+4)^{\sigma}$. On the other hand Lemma 6 shows that

$$m(\Delta_\nu) \leq 2(n+1)(L+1) + 1$$

because the smallest interval of Δ_ν is of level n. Thus

$$n \geq \frac{m(\Delta_\nu)-1}{2(L+1)} - 1,$$

and (4.19) shows that

$$e_{max}^{(\nu)} \leq C\left(\frac{1}{2^{\sigma}}\right)^{\frac{m(\Delta_\nu)-1}{2(L+1)} - 1} = C'\left(\frac{1}{2}\right)^{\frac{\sigma}{2(L+1)} m(\Delta_\nu)}$$

for some constant $C' > 0$, independent of ν. Therefore

$$(4.20) \qquad e_\lambda(\Sigma_\nu) \leq C'(m(\Delta_\nu))^{1/\lambda} e^{-\frac{\sigma \ln 2}{2(L+1)} m(\Delta_\nu)}.$$

In Lemma 7 it has been shown that

$$(4.21) \qquad e_\lambda(\Sigma_\nu) \leq C(m(\Delta_\nu))^{1/\lambda} e^{-(\ln \frac{1}{\gamma})\frac{\deg \Sigma_\nu}{m(\Delta_\nu)}}.$$

(4.20) and (4.21) give

$$e_\lambda(\Sigma_\nu) < C(m(\Delta_\nu))^{1/\lambda} e^{-\frac{1}{2}\left[\frac{\sigma \ln 2}{2(L+1)} m(\Delta_\nu) + \ln\frac{1}{\gamma}\frac{\deg(\Sigma_\nu)}{m(\Delta_\nu)}\right]}$$

$$< C(m(\Delta_\nu))^{1/\lambda} e^{-\sqrt{\frac{\ln 2 \ln 1/\gamma}{2(L+1)}} \cdot \sqrt{\sigma \deg(\Sigma_\nu)}} .$$

Noting that $m(\Delta_N) < \deg(\Delta_N)$, the above inequality implied (4.17) by taking $C_2 = \sqrt{\frac{\ln 2 \ln 1/\gamma}{2(L+1)}} - \varepsilon$ (ε small enough so that $C_2 > 0$), and then choosing C_1. ∎

In (A1) we assume $\xi \in [0,1]$. Suppose $\xi \notin [0,1]$ but (A2), (A3) hold, then we have

Theorem 3. Let hypotheses (A2) - (A3), (E1)′ hold with $\xi \notin [0,1]$, $0 < \gamma < 1$, $0 < \theta < 1$, and let $\{\Sigma_\nu\}_{\nu=0}^{\infty}$ be a trajectory starting with $\Sigma_0 = ([0,1], p_0)$, $p_0 > 1$, then the number of intervals $m(\Delta_\nu)$ is bounded by a finite number when $\nu \to \infty$. Therefore

$$E_\lambda(\Sigma_\nu) < C_1 e^{-C_2 \deg(\Sigma_\nu)} \tag{4.22}$$

with C_1, $C_2 > 0$ independent of ν.

Proof. By Lemma 5, since in this case $x \neq \xi$, the algorithm will generate a permanent p-interval containing each $x \in [0,1]$ (these intervals will never be bisected again). Clearly there are only finitely many such intervals. The rest of the part of the theorem follows from Lemma 7. ∎

We now discuss the adaptivity of the algorithm. Recall in Part 2 we obtained for the model problem that the optimal rate of convergence in the

energy norm of arbitrary mesh-degree combinations (the pairs) when ξ [0,1] is bounded below by

$$C(\alpha)\ \frac{1}{\sqrt{N}^{\,\alpha-1/2}}\ [(\sqrt{2}-1)^2]^{\sqrt{(\alpha-1/2)N}}$$

where N is the number of degree of freedom. This is the case that the local error function was given by

$$E([a,b],p) \;=\; \inf_{v\in \mathbf{P}_{p-1}} \|e'\|_{L_2(a,b)}$$

and $\lambda = 2$, $\sigma = \alpha - 1/2$. We see that this rate of convergence is of the form $C_1 e^{-C_2\sqrt{\sigma N}}$ with $C_2 = \ln[(\sqrt{2}-1)^{-2}]$.

It can be shown under the assumptions of Theorem 2 (with certain condition on ϕ) that there are constants $\hat{C}_1$, $\hat{C}_2 > 0$

$$\Phi(\sigma,N) \;=\; \hat{C}_1\, e^{-\hat{C}_2\sqrt{\sigma N}} \tag{4.23}$$

is the best possible estimate. Therefore we define a performance measure μ_e as follows:

If there are constants $C > 0$, $\rho > 1$, such that

$$E_\lambda(\Sigma_\nu) \;\le\; C[\Phi(\sigma,\deg(\Sigma_\nu))]^{1/\rho}$$

for $\nu = 0,1,2,\ldots,$ then $\mu_e = 1$, otherwise $\mu_e = 0$. We then can say:

Theorem 4. Under the conditions of Theorem 2, the algorithm is adaptive with respect to the performance measure μ_e.

Remark 6. The notion of the adaptivity (as the optimality of the feedback) can be defined in various ways here. It directly relates to the question of comparison of feedback algorithms. In [10] [11] it is shown that if one considers only the worst case problem from a class F, then for many classes F the nonadaptive approach is as good as any adaptive one.

In [12] [13] [4] the optimality is defined asymptotically (for high accuracy and the performance of a trajectory created by an algorithm (for every particular problem) is compared with the performance of the best trajectory (for a given particular problem). In [4] the set F of problems is characterized for which as class of feedback algorithm create trajectories with comparable performance as the theoretically best trajectory and hence the feedback algorithm is an adaptive one.

In this paper we judge the algorithm how it performs with respect to a worst case in a narrow class of solutions having a singularity of the type x^{α} inside or outside of the interval I. It is clear that not a single non-feedback algorithm can perform better than our feedback algorithm for this class of solutions.

Remark 7. It is possible to obtain the results of this section only on the base of the hypotheses (A1)′ - (A3)′ and (E1)′ of the local error indicator without the assumptions of local error function E(I,p). In fact, the only statement in the section which required the property of E(I,p) was

$$\lim_{\nu\to\infty} \max_{I_i^{(\nu)}\in\Delta_\nu} e(I_i^{(\nu)},p_i^{(\nu)}) = 0 \tag{4.24}$$

where Δ_ν is the mesh in the trajectory $\{\Sigma_\nu\}_{\nu=0}^{\infty}$.

Now we prove this directly based on (A1)′ - (A3)′ and (E1)′.

Let $I_c^{(\nu)} \in \Delta_\nu$ be the critical interval, and $x_c^{(\nu)}$ be the middle point of $I_c^{(\nu)}$. Since $0 < x_c^{(\nu)} < 1$, there is a subsequence $\{\nu'\} \subseteq \{\nu\}$

$$\lim_{\nu' \to \infty} x_c^{(\nu')} = x_c.$$

Also let $h_\nu = |I_c^{(\nu)}|$, since $0 < h_\nu < 1$, we can assume for the same subsequence $\{\nu'\}$

$$\lim_{\nu' \to \infty} h_{\nu'} = h.$$

If $h > 0$, there can be only finitely many $I_c^{(\nu')}$ which are different. Therefore, there is ν_0, if $\nu' > c_0$, then $I_c^{(\nu')} = \hat{I}_c$ will be fixed as a permanent p-interval, therefore $p_c^{(\nu')} \to \infty$ and the algorithm gives

$$e(\hat{I}_c, p_c^{(\nu')}) < e(\hat{I}_c, p_c^{(\nu_0)}) \cdot \gamma^{p_c^{(\nu')} - p_c^{(\nu_0)}} \to 0$$

as $\nu' \to \infty$. By (E1)$'$ the maximal local error indicator is non-increasing, therefore (4.29) holds.

We now assume $h = 0$. First, observe in this case we cannot have $x_c \neq \xi$. Form (A2)$'$ and (A3)$'$, the argument in proving Lemma 4 shows that if an interval does not contain ξ, then it cannot be bisected infinitely many times. Thus there are no invervals $I^{(\nu)}$ with $\mathrm{dist}(\xi, I^{(\nu)}) > \varepsilon > 0$ and $|I^{(\nu)}| \to 0$.

This shows we can assume $x_c = \xi$. For each a $I_c^{(\nu')}$, let the interval $\hat{I}^{(\nu')}$ contain both ξ and $I_c^{(\nu')}$. Clearly, we can choose $\hat{I}^{(\nu')}$ to be such that

$$|\hat{I}^{(\nu')}| < |\xi - x_c^{(\nu')}| + h_{\nu'} \to 0 \qquad (\text{as } \nu' \to 0).$$

Therefore by (A1)′ and (E1)

$$e(I_c^{(\nu')}, p_c^{(\nu')}) < e(\hat{I}^{(\nu')}, 1)$$

$$< C_0 |\hat{I}^{(\nu')}|^{\sigma} \to 0.$$

This completes the proof of (4.29). ∎

As a last remark we indicate that the above results are all valid if we increase the degree uniformly on all intervals. In fact, the proofs are concerned with the worst possible degree distribution made by our algorithm. If, instead only increasing the degree on some p-intervals as described in Section 2, we increase all degrees by 1 whenever there is p-interval on which degree is supposed to be increased (cf. Definition 8), then we will obtain an adaptive h-p version algorithm which produces uniform degree vector, and this modification does not cause any change in our original proof.

The case having uniform degree vector is important because it is much easier to make implementation in 2 and 3-dimensional case.

5. NUMERICAL RESULTS

Table 1 and Figure 1 are the numerical results obtained by using the adaptive h-p version algorithm described in Section 2. The problem is the model problem

$$-u'' = f$$

$$u(0) = u(1) = 0$$

with the solution

$$u(x) \;=\; (x-\xi)_+^{\sigma} - (1-\xi)^{\alpha}\cdot x - (-\xi)_+^{\alpha}(1-x)$$

and we compute for the case $\alpha = 0.7$, $\alpha = 1.1$, $\xi = 0$, $\xi = 0.3$, respectively. The local error indicators are exact local error of the finite element solution. According to the theory, we will obtain an exponential rate of error reduction.

$$(5.1) \qquad E_N \approx Cr^{-\kappa\sqrt{N}}$$

where N is the number of degrees of freedom and $r = 10$.

We use linear regression to find the constants C and κ. Comparing with the theoretic values κ' for the h-p extension with geometric mesh and linear degree vector when $q_{opt} = 0.1715$ (the optimal one), $q = 0.5$ with corresponding optimal s, we obtain the following table:

TABLE 1

α	ξ	C	κ	$\kappa/\sqrt{\alpha-1/2}$	κ'		$\kappa'/\sqrt{\alpha-1/2}$	
					q_{opt}=1715	q=0.5	q=0.1715	q=0.5
0.7	0	2.458	0.3174	0.7097	0.3424	0.3036	0.7656	0.6789
	0.3	6.280	0.3566	0.7974				
1.1	0	0.6106	0.5563	0.7181	0.5930	0.5259		
	0.3	6.115	0.6642	0.8575				

Figure 5.1 shows that the error reduction curves are near to straight lines in the $\sqrt{N} \sim \log\|e\|_E$ scaled graph, as expected. The slopes shown in Fig. 5.1 are the theoretical ones for q_{opt} and $q = 0.5$.

For the implementation of the algorithm, as mentioned before, the assumptions of the theory are satisfied for the model problem, $-u'' = f$,

when the solution can be obtained on each mesh interval separately. In the general case, the local errors are affected by global error and the assumptions are not to be satisfied. Furthermore, there is a problem in finding an effective local error estimator for large mesh intervals and high degrees. Although there are difficulties in both theoretical and practical

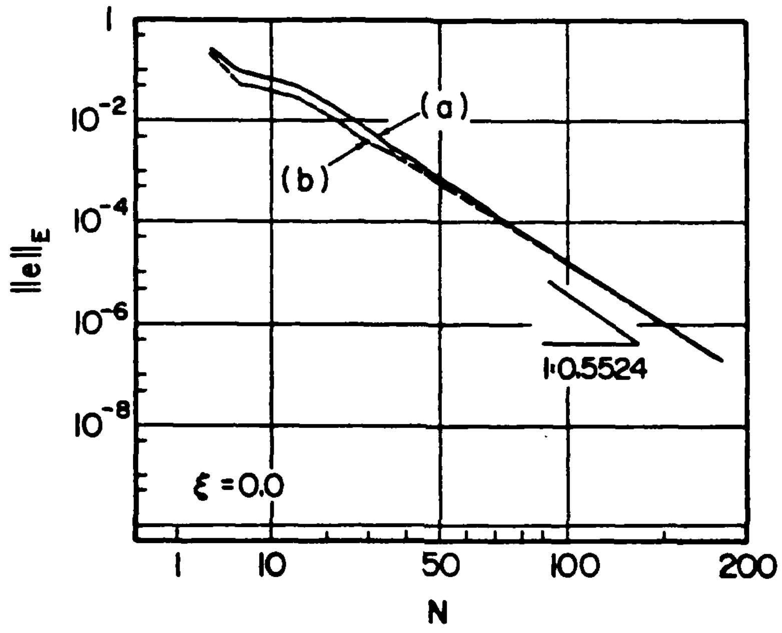

Figure 5.1.

aspects of the adaptive h-p version of the FEM, a program was written for the one-dimensional two-point boundary value problem:

$$-(a(x)u')' + b(x)u = f(x) \qquad x \in (0.1)$$

$$u(0) = u(1) = 0$$

Figures 5.2(a), (b) are the results obtained by using the adaptive h-p version FEM program to solve the following problem

$$-u'' + xu = f$$

$$u(0) = u(1) = 0$$

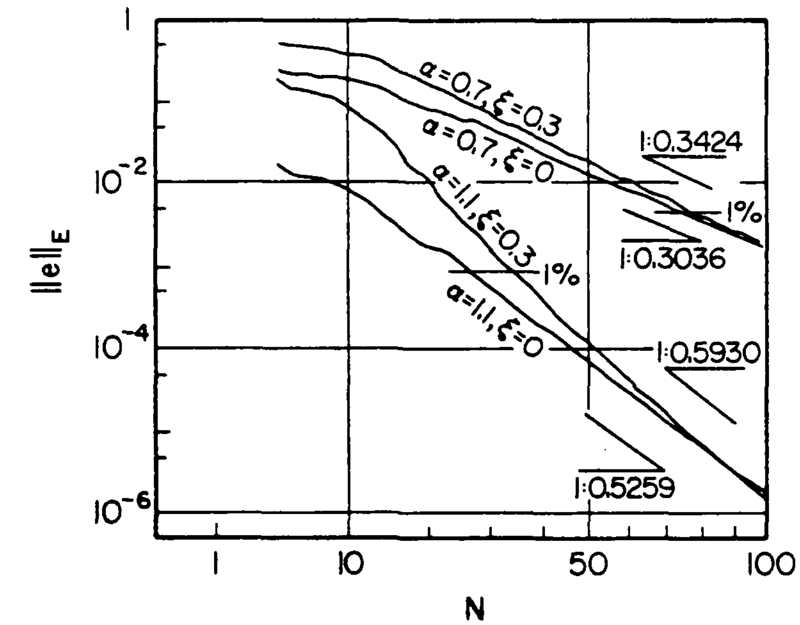

Figure 5.2(a).

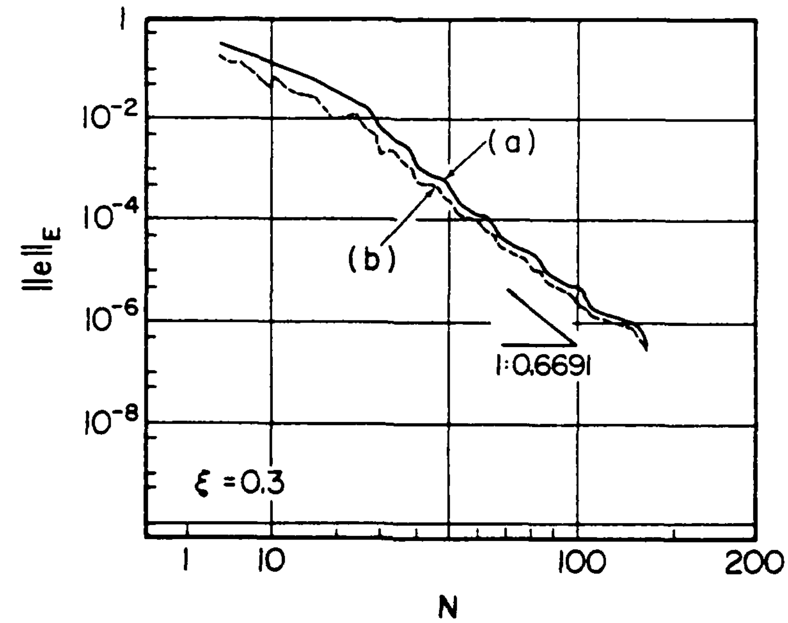

Figure 5.2(b).

with the solution

$$u(x) = (x-\xi)_+^{\alpha} - (1-\xi)^{\alpha}x$$

for $\alpha = 1.1$, $\xi = 0$ and $\xi = 0.3$. The graph is in $\sqrt{N} - \log\|e\|_E$ scale, (a) shows the error. (b) (the dotted curves) are curves of estimated error (by the global error estimates).

We also use linear regression to obtain the constant κ defined in formula (5.1), which is shown on the figures.

Our results show the program also performs very well on this problem. The global error estimator is very reliable. When $\xi = 0.3$ the singular point will never be a nodal point during bisection. In this case the curve, although oscillating, still gives the expected exponential rate, with the rate of convergence and the error itself better than when $\xi \neq 0$.

The program basically agrees with the algorithm as described in Section 2. There are nevertheless some different features in the program:

1) We not only increase the polynomial degrees but also allow lowering them. The reason for doing this is to make the local error more equilibrated so that we can avoid unnecessarily increasing the total number of degrees of freedom. This is done in the following way.

For each $\nu = 1,2,\ldots$, we let $\rho_0 << \gamma$ be given,

$$\rho_\nu = \min\{\rho_{\nu-1}, R^{(\nu)}\}, \qquad \nu = 1,2,\ldots \tag{5.2}$$

with

$$R^{(\nu)} = \min\{R \mid R = \frac{e(I,p+1)}{e(I,p)},\ (I,p) \quad \Sigma_\nu\}.$$

If $e_j^{(\nu)} = e(I_j^{(\nu)}, p_j^{(\nu)}) < \rho_\nu \cdot e_{max}^{(\nu)}$ $\left(e_{max}^{(\nu)} = \max\limits_{1 \le j \le m(\Delta_\nu)} \{e_j^{(\nu)}\}\right)$, then the degree $p_j^{(\nu)}$ of $I_j^{(\nu)}$ will be lowered by 1 (but keep $p_j^{(\nu)} \geq 1$).

In general, lowering the degree could be dangerous. Consider the following situation: Let e_1, e_2 be two local error indicators, and $\rho = \rho_\nu$ is given a priori. Suppose that the two intervals are of p-type and

$$e_1 < \rho e_{max}^{(\nu)}$$

$$e_2 = e_{max}^{(\nu)}.$$

By our algorithm, the degree of first interval will be lowered by 1 and its local error indicator will change to $e_1' > e_1$. Meanwhile the degree of second interval will be raised by 1 and its local error indicator will change to $e_2' < e_2$. However, it may happen that in the next step we have

$$e_1' = e_{max}^{(\nu+1)}$$

$$e'_2 < \rho e_{max}^{(\nu+1)}.$$

Then everything will be back to the original state and the program will run into a viscious circle.

Observe that the conditions for this to happen are

$$e_1 < \rho e_2, \qquad e_2' < \rho e_1'$$

thus

$$R_1 = \frac{e_1}{e_1'} < \rho \frac{e_2}{e_1'} = \rho \frac{e_2}{e_2'} \frac{e_2'}{e_1'} < \rho^2 \frac{1}{R_2}$$

where $R_2 = e_2'/e_2$. Therefore $\rho^2 > R_1 R_2$. If we choose ρ as in (5.2), the vicious circle can be prevented.

2) To find a good local error indicator is not a simple problem,

since the h-p version program produces large elements and high degrees. We are using the following method:

On the interval I (which has degree p), the finite element solution can be written as

$$u_s(x) = C_0 q_{1,0}^{(I)}(x) + C_1 q_{1,1}^{(I)}(x) + C_2 q_2^{(I)}(x) + \cdots + C_p q_p^{(I)}(x)$$

where $q_{1,0}^{(I)}(x)$ and $q_{1,1}^{(I)}(x)$ are linear such that

$$q_{1,0}^{(I)}(x_{i-1}) = 1, \qquad q_{1,0}^{(I)}(x_i) = 0$$

$$q_{1,1}^{(I)}(x_{i-1}) = 0, \qquad q_{1,1}^{(I)}(x_i) = 1$$

and $q_i^{(I)}(x)$ $(i > 2)$ are integrals of Legendre polynomials (transformed to $I = [x_{i-1}, x_i]$). We predict C_{p+1} by solving a local stiffness matrix, and the local error indicators are defined to be

$$e(I,p) = \{\tfrac{1}{2} (\|C_{p-1} q_{p-1}^{(I)}\|_E^2 + \|C_p q_p^{(I)}\|_E^2)\}^{1/2} .$$

There is no theoretical analysis available which shows how good is this error indicator. Our numerical computation shows that for our examples it performes well, but on the interval with singularity in it, this error indicator is low quality.

6. THE PERFORMANCE OF VARIOUS VERSIONS OF THE FEM, THE CONCLUSIONS

In this section we will compare and summarize the performances of all versions of the finite element method in a concrete setting of an example.

Let us consider the problem

$$-u'' = f$$

$$u(0) = u(1) = 0$$

with the exact solution

$$u(x) = x^{\alpha} - x, \qquad \alpha = 1.7$$

having a relatively strong singularity at the origin.

As before, we are interested in the performance measured by the energy norm of the error. The graphs plotted in the double logarithmic scale will show the dependence of the error on the number N of degrees of freedom.

Fig. 6.1 shows the performance of the h-version $(p = 1,2)$ for the uniform, the optimal radical and feedback h-version. For comparison, we also show the performance of the optimal h-p version (i.e., the geometric

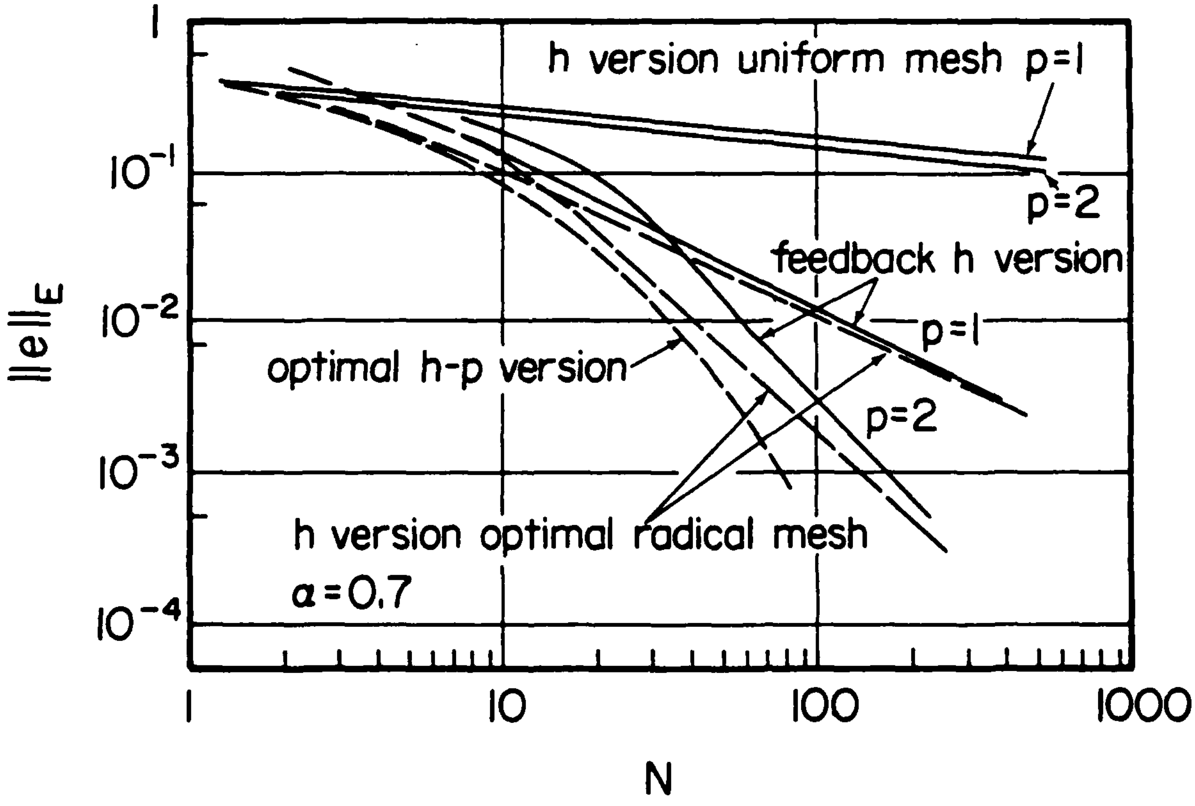

Figure 6.1.

mesh with the ratio $g = (\sqrt{2} - 1)^2 = .1715$ combined with a linear slope of degrees $s = 2\alpha - 1 = .4$).

The figure clearly indicates that the h-version with uniform mesh is not acceptable. The h-version with optimal mesh performs relatively well but strong refinement could cause round-off problems. For $p = 2$ the relative accuracy of 1% is achieved with $N \approx 40$ and the ratio of the sizes of the maximal and minimal elements is 10^{15}. The h-p version requires $N \approx 35$, maximal degree 5 and the ratio of the series of elements 10^9 for achieving the same accuracy of 1%. Fig. 6.1 also shows the performance of the feedback h-version for $p = 1$ and $p = 2$. The feedback method is here adaptive with respect to the rate of convergence u. The rate is the same as the rate of the h-version with optimal mesh. (The feedback approach is more expensive than the computation with a-priori given radical mesh. Nevertheless the cost is not too high.) The figure shows clearly that when higher accuracy is required, then the difference between the performances of various versions increases.

Figure 6.2 compares the performance of the h, p and h-p versions. It shows the performance of the p-version (uniform $p \leq 10$ with the geometric mesh ($q = 0.15$) and $m = 2,5,10$ elements. We also show the performance of the p-version (with the same number of elements m) when the degrees p are chosen in a feedback way. For $m = 1$ and $m = 5$ the p-version fails to achieve accuracy of 1%. If $m = 10$ then the size of the smallest element is of order 10^{-8} and the accuracy of 1% is achieved for $p = 4$. We see that here (i.e., for $m = 10$) the p-version performs in the certain range of accuracy similarly as the hp-version. This clearly indicates the importance of the selection of a proper mesh. We also see here the typical shape of the curve (the S-shape) when in the

first phase the error decreases exponentially and in the second phase algebraicaly with the rate $N^{-2(\alpha - 1/2)}$.

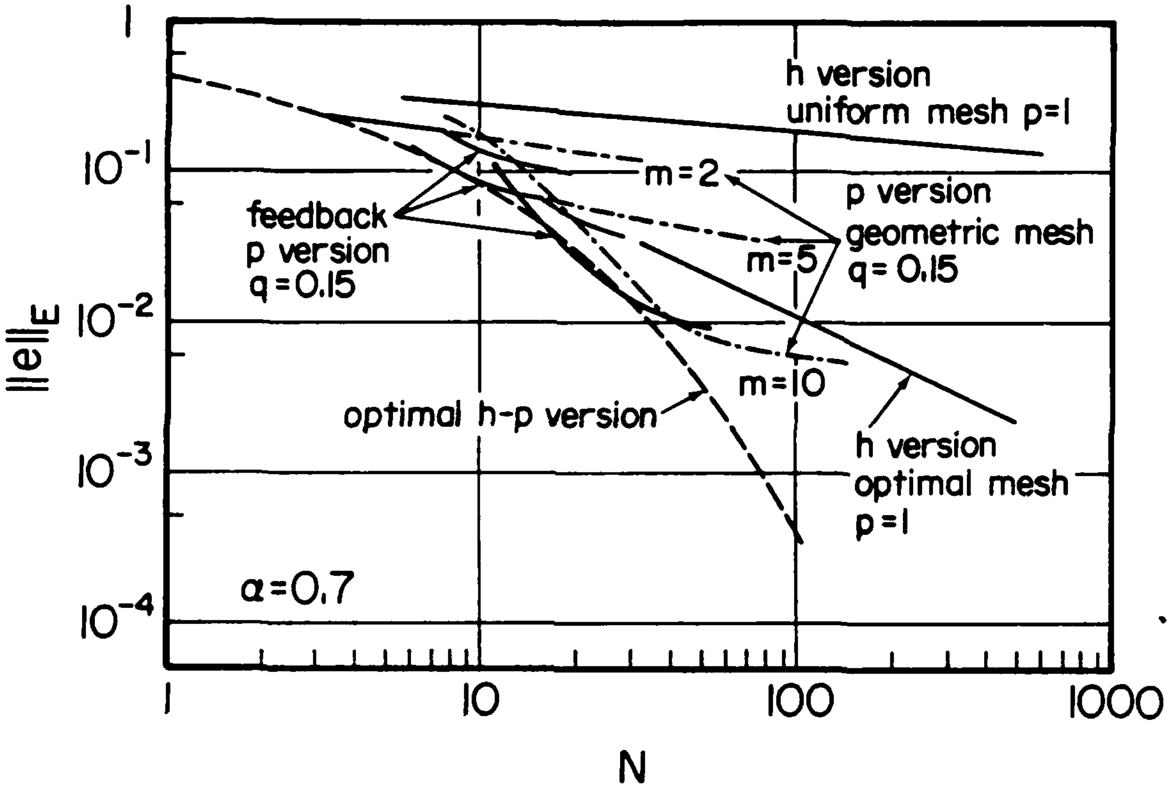

Figure 6.2.

Fig. 6.3 shows the performance of the optimal h-p version with optimal mesh and uniform and nonuniform (optimal) distributions of the degrees of elements. We see the exponential rate of convergence in both cases. The accuracy of 1% is achieved with $N = 35$ for the optimal non-uniform p-distribution and $N = 50$ for the optimal uniform distribution of the degrees of the elements. Fig. 6.3 also shows the performance of the feedback h-p version. We see the same rate although the error is slightly larger. It is clear that the h-p version is expecially effective when higher accuracy is required.

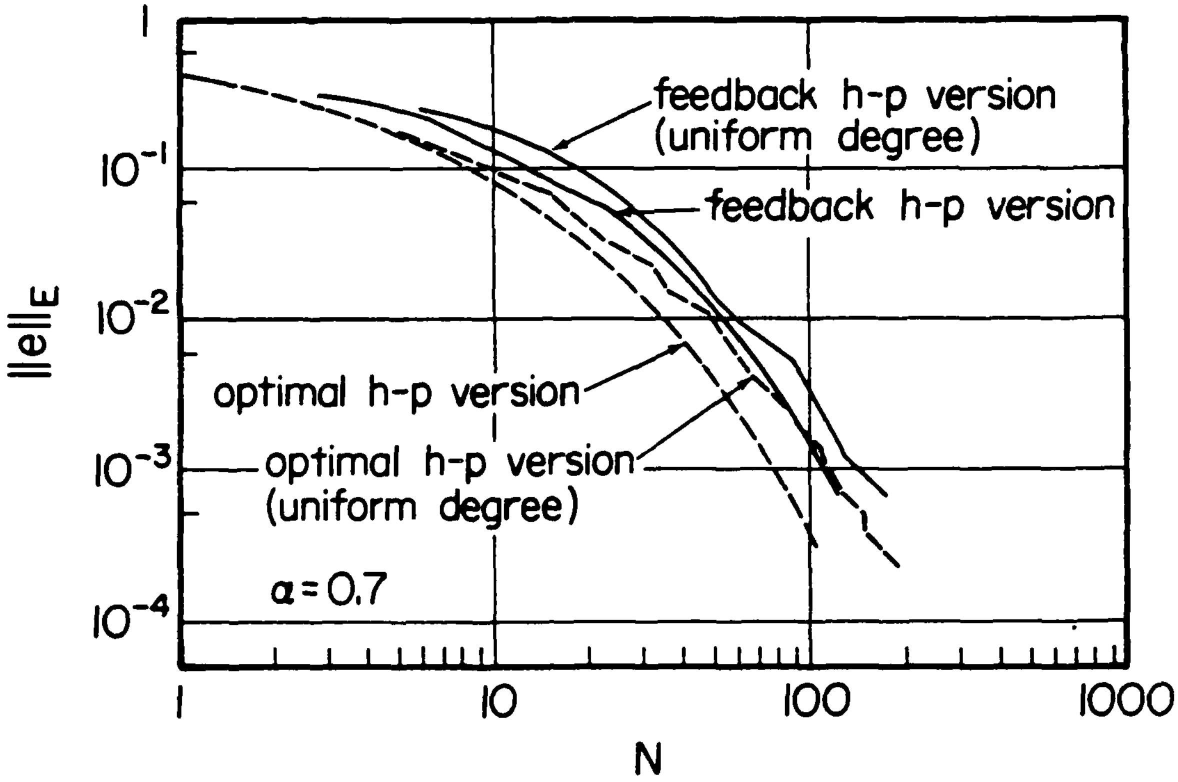

Figure 6.3.

In summary we conclude

1) A uniform mesh cannot produce accurate results for a reasonable cost if the solution has a singular behavior.

2) The proper selection of the mesh is essential for the performance. The mesh can be constructed a-priori if the structure of the solution is known or it can be constructed in a feedback way. The underrefinement of the mesh has to be avoided. The overrefinement does not influence too negatively the performance.

3) The higher degree elements with properly designed mesh perform better than elements of low degrees for both smooth and nonsmooth solution. If high accuracy is required then, especially, the high order element perform well.

4) The p-version is in practical computations effective provided the mesh is properly designed and the required accuracy is achieved in the exponential phase. If the mesh is not properly designed then the p-version does not perform well for singular solution although better than the h-version with uniform mesh.

5) The feedback methods can be designed so that they are adaptive with respect to the convergence and to the convergence rate u. They perform comparably as the optimal meshes.

Although our conclusions are based on the one dimensional case, our results and computational experience related to the two dimensional problems indicate that the conclusion are valid also in two dimension case.

Let us mention that we did not address various aspects of computational complexity as number of operations, data flow problems, etc. These aspects will be addressed in detail in [12] in the two dimensional setting.

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